ISASNS-HEP-94/86

On the Avalanche-finiteness of Abelian Sandpiles

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(February 5, 2008)

Abstract

We prove a necessary and sufficient condition for an Abelian Sandpile Model (ASM) to be avalanche-finite, namely: all unstable states of the system can be brought back to stability in finite number of topplings. The method is also computationally feasible since it involves no greater than $\mathrm{O}\left(N^3\right)$ arithmetic computations where N is the total number of sites of the system.

Key words: Abelian sandpile model; avalanche-finiteness; self-organized criticality

PACS numbers: 64.60.Ht, 02.40.Dr, 05.40.+j, 05.50.+q

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The Abelian Sandpile Model (ASM), whose mathematical structure is first studied extensively by Dhar [1], is one of the few class of models of self-organized criticality in which a lot of interesting physical properties can be found analytically. The model consists of a finite number of sites labeled by an index set I. For each site $i \in I$, we assign an integer h_i called the local height to it. Whenever the local height of a site exceeds a threshold (which is fixed to 0 for simplicity), the site is called unstable and it will transport some of its local heights (or sometimes called particles at that site) to the other sites in the coming timestep according to

$$h_j \longrightarrow h_j - \Delta_{ij} \qquad \text{whenever } h_i > 0. \tag{1}$$

Δ is called the toppling matrix whose elements satisfies

$$\Delta_{ii} > 0 \qquad \forall\, i \in I, \tag{2a}$$

and

$$\Delta_{ij} \leq 0 \qquad \forall\, i \neq j. \tag{2b}$$

Sometimes, we may further require that particles cannot be created in the re-distribution process, which means that

$$\sum_{j \in I} \Delta_{ij} \geq 0 \qquad \forall\, i \in I. \tag{2c}$$

Toppling is repeated until all sites become stable again. The whole process of toppling is collectively known as an avalanche. The system is driven by adding a unit amount of particles onto the sites randomly and uniformly after the system regains stability.

It is, therefore, important to know the condition(s) for which the system relaxes to a stable state in a finite number of topplings. In fact, almost all previous studies of the Abelian Sandpile Model assume that such a relaxation can be taken in a finite number of steps. An Abelian Sandpile is said to be **avalanche-finite** if and only if every unstable system configuration can regain stability in a finite number of steps via toppling. In this paper, we prove a necessary and sufficient condition for avalanche-finiteness. Actually our proof works equally well on integer or real toppling matrices.

A system configuration, stable or not, can be regarded as a point in the space $\mathbb{R}^N$ where N is the total number of sites in the system. Moreover, toppling of particles can be regarded as a translation in $\mathbb{R}^N$. If site i of a system configuration α is unstable, then the system configuration will become $\alpha - \Delta_i$ in the next timestep where Δ_i is the i-th row of the toppling matrix Δ [2,3].

Dhar showed that the order of toppling does not affect the outcome of the final stable state in an avalanche. In this respect, the model is commutative. He then went on to prove that the average number of toppling occurs in site j given that a particle is introduced in site i, G_{ij}, is given by

$$G_{ij} = \left(\Delta^{-1}\right)_{ij} \tag{3}$$

provided that Δ is invertible [1].

Now we are going to prove a necessary and sufficient condition for avalanche-finiteness.

Proposition 1: An Abelian Sandpile Model with toppling matrix Δ is avalanche-finite if and only if there exist $n_i \in \mathbb{Z}^+\ \forall i \in I$ such that $\sum_i n_i \Delta_{ij} > 0$ for all $j \in I$.

Proof: (⇒) Consider the unstable configuration $\alpha = (1, 1, \ldots, 1)$. Avalanche-finiteness implies the existence of $n_i \in \mathbb{Z}^+$ such that $\alpha - \sum_i n_i \Delta_i$ is a stable configuration. As a result, $1 - \sum_i n_i \Delta_{ij} \leq 0$ for all $j \in I$ and hence it is proved.

(⇐) Suppose Δ is not avalanche-finite, then there exists an $\alpha \in \mathbb{R}^N$ which cannot regain stability by toppling in a finite number of steps. Let J be the set of all unstable sites at this moment. Define $\beta^{(1)} = \left(\beta_i^{(1)}\right)_{i\in I}$ by

$$\beta_j^{(1)} = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases} . \tag{4}$$

Then in the next timestep, α will become $\alpha - \beta\Delta$. As the toppling will never stop, we will get an infinite sequence of vectors $\{\beta^{(i)}\}$ by repeating the above process. Now if 1 occurs finitely many times in $\{\beta_j^{(i)}\}_{i\in\mathbb{Z}^+}$, then site j will be stable eventually. Let I' be the set of all sites which become stable in a finite number of steps. Clearly $I \neq I'$ or else the toppling stops in finite number of steps. It is easy to see that $\Delta_{ij} = 0$ for $i \in I \backslash I'$ and $j \in I'$. Besides, Eq. (2b) together with $\sum_{i\in I} n_i \Delta_{ij} > 0$ implies that $\sum_{i\in I\backslash I'} n_i \Delta_{ij} > 0$ for all $j \in I \backslash I'$. But this means the local height for sites in $I \backslash I'$ will be negative after sufficiently long time, which is impossible. □

Proposition 2: An Abelian Sandpile Model with toppling matrix Δ is avalanche-finite if and only if $\det \Delta \neq 0$ and $G_{ij} \equiv (\Delta^{-1})_{ij} \geq 0$ for all $i, j \in I$.

Proof: Let $P_i = \{\sum_{j\neq i} b_j \Delta_j : b_j \geq 0 \ \forall j\}$ be the non-negative span of $\{\Delta_j\}_{j\in I\backslash\{i\}}$, and $C = \{\sum_{j\in I} b_j \Delta_j : b_j \geq 0 \ \forall j\}$ be the cone generated by $\{\Delta_j\}_{j\in I}$. Eq. (2b) tells us that $\bigcup_{i\in I} P_i \cap (\mathbb{R}^+)^N = \emptyset$.

We consider the situation where $\det \Delta \neq 0$ first. As $\{\Delta_j\}_{j\in I}$ is linearly independent, $\bigcup_{i\in I} P_i$, the boundary of C, disconnects $\mathbb{R}^N$. So either (i) $C \supset (\mathbb{R}^+)^N$; or (ii) $C \cap (\mathbb{R}^+)^N = \emptyset$.

Case(i): $\Leftrightarrow (\mathbb{R}^+)^N \Delta^{-1} \subset (\mathbb{R}^+ \cup \{0\})^N$. That is, $(\mathbb{R}^+)^N \Delta^{-1}$ is an open cone in $(\mathbb{R}^+ \cup \{0\})^N$. Therefore, $(\mathbb{R}^+)^N \Delta^{-1}$ contains (infinitely many) positive integer lattice points. Hence result follows from Proposition 1.

Now we consider the remaining possibility where $\det \Delta = 0$. As $\{\Delta_j\}_{j\in I}$ is linearly dependent, one can check that $\bigcup_{i\in I} P_i = C$, hence $C \cap (\mathbb{R}^+)^N = \emptyset$. Thus, for any $n_i \in \mathbb{Z}^+ \ \forall i \in I$, there exists $j \in I$ such that $\sum_i n_i \Delta_{ij} \leq 0$. By Proposition 1, the system is not avalanche-finite. □

With the above two propositions at hand, we are ready to prove our main theorem.

Theorem 1: An Abelian Sandpile Model with toppling matrix Δ satisfying Eqs. (2a) and (2b) is avalanche-finite if and only if $\det \Delta > 0$ and $G_{ij} \equiv (\Delta^{-1})_{ij} \geq 0$ for all $i, j \in I$.

Proof: By Proposition 2, it suffices to show that "the ASM is avalanche-finite implies $\det \Delta > 0$". We prove this claim by induction on N.

Obviously, our theorem holds when $N = 1$. Suppose it is true for $N = k - 1$. Now for $N = k$, Proposition 2 assures that $\det \Delta \neq 0$. Let Δ' be the $(k-1) \times (k-1)$ sub-matrix formed by deleting the k-th row and k-th column of Δ. Using Eq. (2b) and Proposition 1, Δ' is avalanche-finite with $k - 1$ sites. So by induction hypothesis, $C_{kk}(\Delta) = \det \Delta' > 0$ where $C_{kk}(\Delta)$ denotes the (k, k)-th cofactor of Δ. By Proposition 2, $(\det \Delta) \cdot C_{kk}(\Delta) \geq 0$. Therefore, $\det \Delta > 0$. So it is proved. □

Remark 1: For $N = 1, 2, 3$, one can easily check by direct computation that an Abelian Sandpile Model is avalanche-finite if and only if $\det \Delta > 0$.

Remark 2: For $N \geq 4$, $\det \Delta > 0$ alone is not a sufficient condition for avalanche-finiteness. Consider the system with Δ given by

$$\Delta = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \oplus I_{N-4} \tag{5}$$

with $\alpha = (1, 0, 0, \ldots, 0)$. Direct computation verifies that the system is not avalanche-finite.

Finally, if we restrict ourselves to the case where particles cannot be created during the toppling process, a simple criterion for avalanche-finiteness is found. But before we state our finding, we first report two important lemmas.

Lemma 1: $\det \Delta \geq 0$ if Δ satisfies Eqs. (2a)–(2c).

Pf: Consider the elementary row operations of

$$[\text{row } i] \longrightarrow [\text{row } i] - \frac{\Delta_{i1}}{\Delta_{11}}[\text{row } 1], \tag{6}$$

for $i \geq 2$, with resulting matrix $\Delta^{(1)} = \left[\Delta^{(1)}_{ij}\right]$. Now $\Delta^{(1)}_{i1} = 0$ for $i \geq 2$, $\Delta^{(1)}_{ij} \leq 0$ for $i \neq j$, $i \geq 2$. Consider

$$\Delta^{(1)}_{22} = \Delta_{22} - \frac{|\Delta_{21}|\,|\Delta_{12}|}{\Delta_{11}} \geq \Delta_{22} - |\Delta_{21}| = \Delta_{22} + \Delta_{21} \geq 0 \tag{7}$$

with equality holds if and only if $\Delta_{11} = |\Delta_{12}|$ and $\Delta_{22} = |\Delta_{21}|$. Eqs. (2a) and (2b) implies that $\Delta_{1j} = \Delta_{2j} = 0$ for $j \geq 2$. So $\Delta^{(1)}_{22} = 0$ if and only if $\Delta^{(1)}_{2} = \vec{0}$. Then $\det \Delta = 0$.

On the other hand, if $\Delta^{(1)}_{22} > 0$, then $\Delta^{(1)}_{ii} > 0$, $\Delta^{(1)}_{ij} \leq 0$ for $i \neq j$, ≥ 2. Also, it is easy to verify that $\sum_{j>1} \Delta^{(1)}_{ij} \geq 0$. Hence we can repeat our argument on a smaller sub-matrix until either (i) we get to the point where the equality holds in Eq. (7) and hence $\det \Delta = 0$; or (ii) we eventually end up with an upper triangular matrix with positive diagonal elements, and hence $\det \Delta > 0$. □

Lemma 2: $C_{ij}(\Delta) \geq 0$ for all $i, j \in I$ if Δ satisfies Eqs. (2a)–(2c). Here $C_{ij}(\Delta)$ denotes the (i, j)-th cofactor of Δ.

Proof: Note that $C_{ij}(\Delta) = \det \Xi$, where Ξ is the $N \times N$ matrix with elements given by

$$\Xi_{pq} = \begin{cases} 1 & \text{if } p = i \text{ and } q = j \\ 0 & \text{if } p = i \text{ and } q \neq j \\ 0 & \text{if } p \neq i \text{ and } q = j \\ \Delta_{pq} & \text{otherwise} \end{cases}. \tag{8}$$

It is easy to check that the row sums of Ξ are all non-negative except for the j-th row. Let k be an integer greater than $\sum_{r \in I \setminus \{j\}} |\Delta_{jr}|$, and consider the row operation $[\text{row } j] \longrightarrow [\text{row } j] + k[\text{row } i]$.

Then it is obvious that the row sums of this new matrix are all non-negative and hence by Lemma 1, $C_{ij}(\Delta) = \det \Xi \geq 0$. $\square$

Remark 4: Thus if we further require Δ to be invertible, then $G_{ij} \equiv \left(\Delta^{-1}\right)_{ij} \geq 0$ for all $i, j \in I$.

Corollary 1: An Abelian Sandpile Model with toppling matrix Δ satisfying Eqs. (2a)–(2c) is avalanche-finite if and only if $\det \Delta > 0$.

Proof: Direct application of Theorem 1 and Lemma 2. $\square$

In conclusion, we found a necessary and sufficient condition for avalanche-finiteness for Abelian Sandpiles. In the event that particles may be created during the toppling process, avalanche-finiteness can be examined by looking at the determinant and the elements of the inverse of the toppling matrix Δ, a process requiring not greater than $\mathrm{O}\left(N^3\right)$ arithmetic computations. Thus, our method is computationally practical. Furthermore, our finding is consistent with the interpretation that $\left(\Delta^{-1}\right)_{ij}$ is the average number of topplings occurs in site j given a particle is introduced in site i.

If we further restrict ourselves to the case where particle cannot be created in toppling, our avalanche-finiteness testing scheme can be further simplified to a simple test in the sign of $\det \Delta$, which again can be done efficiently in the computational point of view.

ACKNOWLEDGMENTS

This work is supported in part by DOE grant DE-FG02-90ER40542.

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